

# On the uniqueness of solution to the steady Euler equations with perturbations

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## Abstract

In this paper we study the uniqueness property of solutions to the steady incompressible Euler equations with perturbations in  $\mathbb{R}^N$ . Our perturbations include as special cases the Euler equations with a ‘single signed’ nonlinear term, the self-similar Euler equations, and the steady Navier-Stokes equations. For these equations show that suitable decay assumptions at infinity on the solution or its derivatives, imposed by the  $L^q$  conditions imply that the only possible solution is zero.

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## 1 Main theorems

We are concerned on the steady equations on  $\mathbb{R}^N$  with perturbation.

$$\begin{cases} (v \cdot \nabla)v = -\nabla p + \Phi(v), \\ \operatorname{div} v = 0, \end{cases} \quad (1.1)$$

where  $v = v(x) = (v_1(x), \dots, v_N(x))$  is the velocity, and  $p = p(x)$  is the pressure. The function  $\Phi : \mathbb{R}^N \rightarrow \mathbb{R}^N$  defining the perturbation term satisfies

suitable conditions depending on the cases we consider below. We study the vanishing property of the solutions to (1.1). In this paper we consider the three cases of  $\Phi(v)$ . One is case where  $\Phi(v)$  represents a single signed nonlinear function(see below for more precise definition), and the other one is the case where the system (1.1) corresponds to a generalization of the self-similar Euler equations, and finally the case where  $\Phi(v) = \Delta v$ , which corresponds to the steady Navier-Stokes equations.

### 1.1 The case where $\Phi(v) \cdot v$ is single signed

Let us fix  $N \geq 2$ . Here we assume that the continuous function  $\Phi(\cdot) : \mathbb{R}^N \rightarrow \mathbb{R}^N$  satisfies the condition of single signedness:

$$\forall v \in \mathbb{R}^N \text{ either } \Phi(v) \cdot v \geq 0 \text{ or } \Phi(v) \cdot v \leq 0, \quad (1.2)$$

and

$$\Phi(v) \cdot v = 0 \text{ if only if } v = 0. \quad (1.3)$$

For such given  $\Phi$  we consider the system (1.1). Note that when  $\Phi(v) = -v$  the system (1.1)-(1.3) becomes the usual steady Euler equations with a damping term. More generally  $\Phi(v) = G(|v|)v$  with a scalar function  $G(x) > 0$  for  $x > 0$  satisfies (1.2)-(1.3). We observe that the system (1.1)-(1.3) has a trivial solution  $v = 0$ . We will prove that the uniqueness of solution to the system (1.1)-(1.3) under quite mild decay conditions on the solutions. More specifically we will prove the following.

**Theorem 1.1** *Let  $(v, p)$  be a  $C^1(\mathbb{R}^N)$  solution of (1.1)-(1.3). Suppose there exists  $q \in [\frac{3N}{N-1}, \infty)$  such that*

$$v \in L^q(\mathbb{R}^N) \quad \text{and} \quad p \in L^{\frac{q}{2}}(\mathbb{R}^N). \quad (1.4)$$

*Then,  $v = 0$ .*

*Remark 1.1* If  $\Phi(v)$  satisfies an extra condition  $\text{div} \Phi(v(x)) = 0$ , then we do not need to assume  $p \in L^{\frac{q}{2}}(\mathbb{R}^N)$  in (1.4). Since in that case we have the well-known velocity-pressure relation as in the incompressible Euler or the Navier-Stokes equations,

$$p(x) = \sum_{j,k=1}^N R_j R_k (v_j v_k)(x)$$

with the Riesz transform  $R_j$ ,  $j = 1, \dots, N$ , in  $\mathbb{R}^N$  ([8]), and the  $L^{\frac{q}{2}}$  estimate of the pressure follows from the  $L^q$  estimate for the velocity by the Calderon-Zygmund inequality,

$$\|p\|_{L^{\frac{q}{2}}} \leq C \sum_{j,k=1}^N \|R_j R_k v_j v_k\|_{L^{\frac{q}{2}}} \leq C \|v\|_{L^q}^2 \quad 2 < q < \infty. \quad (1.5)$$

## 1.2 The case $\Phi(v) = av + b(x \cdot \nabla)v$ , $ab \neq 0$

In this subsection we fix  $N = 3$ . Let  $a, b$  be given constants such that  $ab \neq 0$ . We study here the system in  $\mathbb{R}^3$ .

$$\begin{cases} (v \cdot \nabla)v = -\nabla p + av + b(x \cdot \nabla)v, \\ \operatorname{div} v = 0. \end{cases} \quad (1.6)$$

In the special case of  $a = -\frac{\alpha}{\alpha+1}$ ,  $b = -\frac{1}{\alpha+1}$  the system (1.6) reduces to the self-similar Euler equations.

$$\begin{cases} \frac{\alpha}{\alpha+1}v + \frac{1}{\alpha+1}(x \cdot \nabla)v + (v \cdot \nabla)v = -\nabla p, \\ \operatorname{div} v = 0. \end{cases} \quad (1.7)$$

The system (1.7) is obtained from the time dependent Euler equations,

$$(E) \begin{cases} \partial_t u + (u \cdot \nabla)u = -\nabla \mathbf{p}, \\ \operatorname{div} u = 0, \end{cases}$$

by the self-similar ansatz,

$$\begin{aligned} u(x, t) &= \frac{1}{(t_* - t)^{\frac{\alpha}{\alpha+1}}} v \left( \frac{x - x_*}{(t_* - t)^{\frac{1}{\alpha+1}}} \right), \\ \mathbf{p}(x, t) &= \frac{1}{(t_* - t)^{\frac{2\alpha}{\alpha+1}}} p \left( \frac{x - x_*}{(t_* - t)^{\frac{1}{\alpha+1}}} \right), \end{aligned}$$

where  $(x_*, t_*)$  is the hypothetical self-similar blow-up space-time point. The question of self-similar blow-up for the Navier-Stokes equations is asked in ([6]), and is answered negatively in [7] for  $v \in L^3(\mathbb{R}^3)$ , and is extended in [9]

for  $v \in L^q(\mathbb{R}^3)$ ,  $q \geq 3$ . Similar problem for the Euler equations is studied in [1, 2]. For  $\alpha < \infty$  with  $\alpha \neq -1$  it is proved in [1] that if a solution to (1.7),  $v \in C^1(\mathbb{R}^3)$ , decaying to zero at infinity, satisfies  $\omega = \text{curl } v \in \cap_{0 < q < q_0} L^q(\mathbb{R}^3)$  for some  $q_0 > 0$ , then  $v = 0$ . In the extreme case  $\alpha = \infty$ , we have (1.7) becomes the system (1.1) with  $\Phi(v) = -v$ .

**Theorem 1.2** *Let  $v$  be a classical solution to (1.6). Suppose there exists  $q_0 > 0$  such that*

$$\|\nabla v\|_{L^\infty} < \infty, \quad \text{and} \quad \omega \in \bigcap_{0 < q < q_0} L^q(\mathbb{R}^3). \quad (1.8)$$

*Then,  $v = \nabla h$  for a harmonic scalar function  $h$  on  $\mathbb{R}^3$ . Thus, if we impose further the condition  $\lim_{|x| \rightarrow \infty} |v(x)| = 0$ , then  $v = 0$ .*

*Remark 1.2* In [1] we used the time dependent Euler equations directly to prove Theorem 1.1, and needed the decay condition for the velocity, since we used the notion of back-to-label map, whose existence is guaranteed for the decaying velocity([4]). In the proof of the above theorem below, however, we work with the stationary system (1.7), and do not use the back-to-label map, and therefore the decay condition for the velocity field is not necessary.

*Remark 1.3* As far as the regularity assumption for the solution  $v$ , what we need in the proof is actually the differentiability almost everywhere, which is guaranteed by the first condition of (1.8).

### 1.3 The case $\Phi(v) = a\Delta v$ , $a \neq 0$

In this subsection we also fix  $N = 3$ . Here we study (1.1) with  $\Phi(v) = a\Delta v$ . In this case without loss of generality we may set  $a = 1$ . In this case the system (1.1) reduces to the steady Navier-Stokes equations in  $\mathbb{R}^3$ .

$$(NS) \begin{cases} (v \cdot \nabla)v = -\nabla p + \Delta v, \\ \text{div } v = 0, \end{cases}$$

We consider here the generalized solutions of the system (NS), satisfying

$$\int_{\mathbb{R}^3} |\nabla v|^2 dx < \infty, \quad (1.9)$$

and

$$\lim_{|x| \rightarrow \infty} v(x) = 0. \quad (1.10)$$

It is well-known that a generalized solution to (NS) belonging to  $W_{loc}^{1,2}(\mathbb{R}^3)$  implies that  $v$  is smooth (see e.g. [5]). Therefore without loss of generality we can assume that our solutions to (NS) satisfying (1.9) are smooth. The uniqueness question, or equivalently the question of Liouville property of solution for the system (NS) under the assumptions (1.9) and (1.10) is a long standing open problem. On the other hand, it is well-known that the uniqueness of solution holds in the class  $L^{\frac{9}{2}}(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3)$ , namely a smooth solution to (NS) satisfying  $v \in L^{\frac{9}{2}}(\mathbb{R}^3)$  and (1.9) is  $v = 0$  (see Theorem 9.7 of [5]). We assume here slightly stronger condition than (1.9), but having the same scaling property, to deduce the uniqueness result. More precisely, we have the following theorem.

**Theorem 1.3** *Let  $v$  be a smooth solution of (NS) satisfying (1.10) and*

$$\int_{\mathbb{R}^3} |\Delta v|^{\frac{6}{5}} dx < \infty. \quad (1.11)$$

*Then,  $v = 0$  on  $\mathbb{R}^3$ .*

*Remark 1.3* Under the assumption (1.10) we have the inequalities with the norms of the *same scaling properties*,

$$\|\nabla v\|_{L^2} \leq C \|D^2 v\|_{L^{\frac{6}{5}}} \leq C \|\Delta v\|_{L^{\frac{6}{5}}} < \infty$$

due to the Sobolev and the Calderon-Zygmund inequalities. Thus, (1.11) implies (1.9). There is no, however, mutual implication relation between Theorem 1.3 and the above mentioned  $L^{\frac{9}{2}}$  result.

*This paper is a modified and extended version of author's preprint [3].*

## 2 Proof of the Main Theorems

**Proof of Theorem 1.1** We denote

$$[f]_+ = \max\{0, f\}, \quad [f]_- = \max\{0, -f\},$$

and

$$D_{\pm} := \left\{ x \in \mathbb{R}^N \mid \left[ p(x) + \frac{1}{2}|v(x)|^2 \right]_{\pm} > 0 \right\}$$

respectively. We introduce the radial cut-off function  $\sigma \in C_0^\infty(\mathbb{R}^N)$  such that

$$\sigma(|x|) = \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 2, \end{cases} \quad (2.1)$$

and  $0 \leq \sigma(x) \leq 1$  for  $1 < |x| < 2$ . Then, for each  $R > 0$ , we define

$$\sigma\left(\frac{|x|}{R}\right) := \sigma_R(|x|) \in C_0^\infty(\mathbb{R}^N).$$

We multiply first equations of (1.1) by  $v$  to obtain

$$v \cdot \Phi(v) = v \cdot \nabla \left( p + \frac{1}{2}|v|^2 \right). \quad (2.2)$$

Next, we multiply (2.2) by  $\left[ p + \frac{1}{2}|v|^2 \right]_+^{\frac{qN-q-3N}{2N}} \sigma_R \operatorname{sign}\{v \cdot \Phi(v)\}$  and integrate over  $\mathbb{R}^N$ , then we have

$$\begin{aligned} & \int_{\mathbb{R}^N} \left[ p + \frac{1}{2}|v|^2 \right]_+^{\frac{qN-q-3N}{2N}} |v \cdot \Phi(v)| \sigma_R dx \\ &= \operatorname{sign}\{v \cdot \Phi(v)\} \int_{\mathbb{R}^N} \left[ p + \frac{1}{2}|v|^2 \right]_+^{\frac{qN-q-3N}{2N}} \sigma_R v \cdot \nabla \left( p + \frac{1}{2}|v|^2 \right) dx \\ &:= I \end{aligned} \quad (2.3)$$

We estimate  $I$  as follows.

$$\begin{aligned} |I| &= \left| \int_{\mathbb{R}^N} \left[ p + \frac{1}{2}|v|^2 \right]_+^{\frac{qN-q-3N}{2N}} \sigma_R v \cdot \nabla \left( p + \frac{1}{2}|v|^2 \right) dx \right| \\ &= \left| \int_{D_+} \left[ p + \frac{1}{2}|v|^2 \right]_+^{\frac{qN-q-3N}{2N}} \sigma_R v \cdot \nabla \left[ p + \frac{1}{2}|v|^2 \right]_+ dx \right| \\ &= \frac{2N}{qN-q-N} \left| \int_{D_+} \sigma_R v \cdot \nabla \left[ p + \frac{1}{2}|v|^2 \right]_+^{\frac{qN-q-N}{2N}} dx \right| \\ &= \frac{2N}{qN-q-N} \left| \int_{D_+} \left[ p + \frac{1}{2}|v|^2 \right]_+^{\frac{qN-q-N}{2N}} v \cdot \nabla \sigma_R dx \right| \end{aligned}$$

$$\begin{aligned}
&\leq \frac{C\|\nabla\sigma\|_{L^\infty}}{R} \left( \int_{\mathbb{R}^N} (|p| + |v|^2)^{\frac{q}{2}} dx \right)^{\frac{qN-q-N}{qN}} \|v\|_{L^q(R \leq |x| \leq 2R)} \times \\
&\quad \times \left( \int_{\{R \leq |x| \leq 2R\}} dx \right)^{\frac{1}{N}} \\
&\leq C\|\nabla\sigma\|_{L^\infty} \left( \|p\|_{L^{\frac{q}{2}}} + \|v\|_{L^q}^2 \right)^{\frac{qN-q-N}{qN}} \|v\|_{L^q(R \leq |x| \leq 2R)} \rightarrow 0 \quad (2.4)
\end{aligned}$$

as  $R \rightarrow \infty$ . Passing  $R \rightarrow \infty$  in (2.3), we obtain

$$\int_{\mathbb{R}^N} \left[ p + \frac{1}{2}|v|^2 \right]_+^{\frac{qN-q-3N}{2N}} |v \cdot \Phi(v)| dx = 0. \quad (2.5)$$

Similarly, multiplying (2.2) by  $\left[ p + \frac{1}{2}|v|^2 \right]_-^{\frac{qN-q-3N}{2N}} \sigma_R$ , and integrate over  $\mathbb{R}^N$ , we deduce by similar estimates to the above,

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left[ p + \frac{1}{2}|v|^2 \right]_-^{\frac{qN-q-3N}{2N}} |v \cdot \Phi(v)| \sigma_R dx \\
&= - \int_{\mathbb{R}^N} \left[ p + \frac{1}{2}|v|^2 \right]_-^{\frac{qN-q-3N}{2N}} \sigma_R v \cdot \nabla \left( p + \frac{1}{2}|v|^2 \right) dx \\
&= \int_{\mathbb{R}^N} \left[ p + \frac{1}{2}|v|^2 \right]_-^{\frac{qN-q-3N}{2N}} \sigma_R v \cdot \nabla \left[ p + \frac{1}{2}|v|^2 \right]_- dx \\
&\leq C\|\nabla\sigma\|_{L^\infty} \left( \|p\|_{L^{\frac{q}{2}}} + \|v\|_{L^q}^2 \right)^{\frac{qN-q-N}{qN}} \|v\|_{L^q(R \leq |x| \leq 2R)} \rightarrow 0 \quad (2.6)
\end{aligned}$$

as  $R \rightarrow \infty$ . Hence,

$$\int_{\mathbb{R}^N} \left[ p + \frac{1}{2}|v|^2 \right]_-^{\frac{qN-q-3N}{2N}} |v \cdot \Phi(v)| dx = 0. \quad (2.7)$$

Let us define

$$\mathcal{S} = \{x \in \mathbb{R}^N \mid v(x) \neq 0\}.$$

Suppose  $\mathcal{S} \neq \emptyset$ . Then, (2.6) and (2.7) together with (1.2)-(1.3) imply

$$\left[ p(x) + \frac{1}{2}|v(x)|^2 \right]_+ = \left[ p(x) + \frac{1}{2}|v(x)|^2 \right]_- = 0 \quad \forall x \in \mathcal{S}.$$

Namely,

$$p(x) + \frac{1}{2}|v(x)|^2 = 0 \quad \forall x \in \mathcal{S}.$$

From (2.2) this implies

$$\Phi(v(x)) \cdot v(x) = 0 \quad \forall x \in \mathcal{S}. \quad (2.8)$$

Considering the conditions on  $\Phi$  in (1.2)-(1.3), we have a contradiction, and therefore we need  $\mathcal{S} = \emptyset$ , namely  $v = 0$  on  $\mathbb{R}^N$ .  $\square$

**Proof of Theorem 1.2** We first observe that from the calculus identity

$$v(x) = v(0) + \int_0^1 \partial_s v(sx) ds = v(0) + \int_0^1 x \cdot \nabla v(sx) ds,$$

we have  $|v(x)| \leq |v(0)| + |x| \|\nabla v\|_{L^\infty} \leq C(1 + |x|) \|\nabla v\|_{L^\infty}$ , and

$$\sup_{x \in \mathbb{R}^3} \frac{|v(x)|}{1 + |x|} \leq C \|\nabla v\|_{L^\infty}. \quad (2.9)$$

We consider the vorticity equation of (1.6),

$$-(a+b)\omega - b(x \cdot \nabla)\omega + (v \cdot \nabla)\omega = (\omega \cdot \nabla)v. \quad (2.10)$$

We take  $L^2(\mathbb{R}^3)$  inner product (2.10) by  $|\omega|^{q-2}\omega\sigma_R$ , then after integration by part, we have

$$\begin{aligned} & -(a+b)\|\omega\sigma_R\|_{L^q}^q + \frac{3b}{q}\|\omega\sigma_R\|_{L^q}^q - \int_{\mathbb{R}^3} \xi \cdot \nabla v \cdot \xi |\omega|^q \sigma_R dx \\ & = b \int_{\mathbb{R}^3} |\omega|^q (x \cdot \nabla) \sigma_R dx + \int_{\mathbb{R}^3} |\omega|^q (v \cdot \nabla) \sigma_R dx \\ & := I + J. \end{aligned} \quad (2.11)$$

We estimate  $I$  and  $J$  easily as follows.

$$|I| \leq \frac{b}{R} \int_{\{R \leq |x| \leq 2R\}} |\omega|^q |x| |\nabla \sigma| dx \leq b \|\nabla \sigma\|_{L^\infty} \|\omega\|_{L^p(R \leq |x| \leq 2R)}^q \rightarrow 0$$

as  $R \rightarrow \infty$ .

$$\begin{aligned} |J| & \leq \frac{1}{R} \int_{\{R \leq |x| \leq 2R\}} |\omega|^q |v| |\nabla \sigma| dx \leq \frac{1+2R}{R} \int_{\{R \leq |x| \leq 2R\}} \frac{|v(x)|}{1+|x|} |\omega|^q |\nabla \sigma| dx \\ & \leq \frac{1+2R}{R} \|\nabla \sigma\|_{L^\infty} \|\nabla v\|_{L^\infty} \|\omega\|_{L^p(R \leq |x| \leq 2R)}^q \rightarrow 0 \end{aligned}$$



as  $R \rightarrow \infty$ , where we used (2.9). Therefore, passing  $R \rightarrow \infty$  in (2.11), we obtain

$$-\|\nabla v\|_{L^\infty}\|\omega\|_{L^q}^q \leq \left(a + b - \frac{3b}{q}\right)\|\omega\|_{L^q}^q \leq \|\nabla v\|_{L^\infty}\|\omega\|_{L^q}^q. \quad (2.12)$$

Suppose  $\omega \neq 0$ , then we will derive a contradiction. If  $\|\omega\|_{L^q} \neq 0$ , we can divide (2.12) by  $\|\omega\|_{L^q}^q$  to have

$$-\|\nabla v\|_{L^\infty} \leq \left(a + b - \frac{3b}{q}\right) \leq \|\nabla v\|_{L^\infty}, \quad (2.13)$$

which holds for all  $q \in (0, q_0)$ . Since  $b \neq 0$ , passing  $q \downarrow 0$  in (2.13), we obtain desired contradiction. Therefore  $\omega = \operatorname{curl} v = 0$ . This, together with  $\operatorname{div} v = 0$ , provides us with the fact that  $v = \nabla h$  for a scalar harmonic function  $h$  on  $\mathbb{R}^3$ .  $\square$

**Proof of Theorem 1.3** Under the assumption (1.11) and Remark 1.1, Theorem IX.6.1 of [5] implies that

$$\lim_{|x| \rightarrow \infty} |p(x) - p_1| = 0. \quad (2.14)$$

for a constant  $p_1$ . Therefore, if we set

$$Q(x) := \frac{1}{2}|v(x)|^2 + p(x) - p_1,$$

then

$$\lim_{|x| \rightarrow \infty} |Q(x)| = 0. \quad (2.15)$$

As before we denote  $[f]_+ = \max\{0, f\}$ ,  $[f]_- = \max\{0, -f\}$ . Given  $\varepsilon > 0$ , we define

$$\begin{aligned} D_+^\varepsilon &= \left\{x \in \mathbb{R}^3 \mid [Q(x) - \varepsilon]_+ > 0\right\}, \\ D_-^\varepsilon &= \left\{x \in \mathbb{R}^3 \mid [Q(x) + \varepsilon]_- > 0\right\}. \end{aligned}$$

respectively. Note that (2.15) implies that  $D_\pm^\varepsilon$  are bounded sets in  $\mathbb{R}^3$ . Moreover,

$$Q \mp \varepsilon = 0 \quad \text{on} \quad \partial D_\pm^\varepsilon \quad (2.16)$$

respectively. Also, thanks to the Sard theorem combined with the implicit function theorem  $\partial D_{\pm}^{\varepsilon}$ 's are smooth level surfaces in  $\mathbb{R}^3$  except the values of  $\varepsilon > 0$ , having the zero Lebesgue measure, which corresponds to the critical values of  $z = Q(x)$ . It is understood that our values of  $\varepsilon$  below avoids these exceptional ones. We write the system (NS) in the form,

$$-v \times \operatorname{curl} v = -\nabla Q + \Delta v. \quad (2.17)$$

Let us multiply (2.17) by  $v [Q - \varepsilon]_+$ , and integrate it over  $\mathbb{R}^3$ . Then, since  $v \times \operatorname{curl} v \cdot v = 0$ , we have

$$\begin{aligned} 0 &= - \int_{\mathbb{R}^3} [Q - \varepsilon]_+ v \cdot \nabla (Q - \varepsilon) dx + \int_{\mathbb{R}^3} [Q - \varepsilon]_+ v \cdot \Delta v dx \\ &:= I_1 + I_2. \end{aligned} \quad (2.18)$$

Integrating by parts, using (2.16), we obtain

$$I_1 = - \int_{D_+^{\varepsilon}} (Q - \varepsilon) v \cdot \nabla (Q - \varepsilon) dx = -\frac{1}{2} \int_{D_+^{\varepsilon}} v \cdot \nabla (Q - \varepsilon)^2 dx = 0$$

Using

$$v \cdot \Delta v = \Delta \left( \frac{1}{2} |v|^2 \right) - |\nabla v|^2, \quad (2.19)$$

and the well-known formula for the Navier-Stokes equations,

$$\Delta p = |\omega|^2 - |\nabla v|^2, \quad (2.20)$$

we have

$$\begin{aligned} I_2 &= - \int_{\mathbb{R}^3} |\nabla v|^2 [Q - \varepsilon]_+ dx + \int_{\mathbb{R}^3} \Delta \left( \frac{1}{2} |v|^2 \right) [Q - \varepsilon]_+ dx \\ &= - \int_{\mathbb{R}^3} |\omega|^2 [Q - \varepsilon]_+ dx + \int_{\mathbb{R}^3} \Delta (Q - \varepsilon) [Q - \varepsilon]_+ dx \\ &:= J_1 + J_2. \end{aligned} \quad (2.21)$$

Integrating by parts, we transform  $J_2$  into

$$J_2 = \int_{D_+^{\varepsilon}} \Delta (Q - \varepsilon) (Q - \varepsilon) dx = - \int_{D_+^{\varepsilon}} |\nabla (Q - \varepsilon)|^2 dx. \quad (2.22)$$

Thus, the derivations (2.18)-(2.22) lead us to

$$0 = \int_{D_+^\varepsilon} |\omega|^2 |Q - \varepsilon| \, dx + \int_{D_+^\varepsilon} |\nabla (Q - \varepsilon)|^2 \, dx \quad (2.23)$$

for all  $\varepsilon > 0$ . The vanishing of the second term of (2.23) implies

$$[Q - \varepsilon]_+ = C_0 \quad \text{on} \quad D_+^\varepsilon$$

for a constant  $C_0$ . From the fact (2.16) we have  $C_0 = 0$ , and  $[Q - \varepsilon]_+ = 0$  on  $\mathbb{R}^3$ , which holds for all  $\varepsilon > 0$ . Hence,

$$[Q]_+ = 0 \quad \text{on} \quad \mathbb{R}^3. \quad (2.24)$$

This shows that  $Q \leq 0$  on  $\mathbb{R}^3$ . Suppose  $Q = 0$  on  $\mathbb{R}^3$ . Then, from (2.17), we have  $v \cdot \Delta v = 0$  on  $\mathbb{R}^3$ . Hence,

$$\Delta p = -\frac{1}{2} \Delta |v|^2 = -v \cdot \Delta v - |\nabla v|^2 = -|\nabla v|^2.$$

Comparing this with (2.20), we have  $\omega = 0$ . Combining this with  $\operatorname{div} v = 0$ , we find that  $v$  is a harmonic function in  $\mathbb{R}^3$ . Thus, by (1.10) and the Liouville theorem for the harmonic function,  $v = 0$ . Hence, without loss of generality, we may assume

$$0 > \inf_{x \in \mathbb{R}^3} Q(x) := -\delta_0.$$

Given  $\delta \in (0, \delta_0)$ , we multiply (2.17) by  $v [Q + \varepsilon]_-^\delta$ , and integrate it over  $\mathbb{R}^3$ . Then, similarly to the above we have

$$\begin{aligned} 0 &= - \int_{\mathbb{R}^3} [Q + \varepsilon]_-^\delta v \cdot \nabla (Q + \varepsilon) \, dx + \int_{\mathbb{R}^3} [Q + \varepsilon]_-^\delta v \cdot \Delta v \, dx \\ &:= I'_1 + I'_2. \end{aligned} \quad (2.25)$$

Observing  $Q(x) + \varepsilon = -[Q(x) + \varepsilon]_-$  for all  $x \in D_-^\varepsilon$ , integrating by part, we obtain

$$\begin{aligned} I'_1 &= \int_{D_-^\varepsilon} [Q + \varepsilon]_-^\delta v \cdot \nabla [Q + \varepsilon]_- \, dx \\ &= \frac{1}{\delta + 1} \int_{D_-^\varepsilon} v \cdot \nabla [Q + \varepsilon]_-^{\delta+1} \, dx = 0. \end{aligned}$$

Thus, using (2.19), we have

$$0 = - \int_{D_-^\varepsilon} |\nabla v|^2 [Q + \varepsilon]_-^\delta dx + \frac{1}{2} \int_{D_-^\varepsilon} [Q + \varepsilon]_-^\delta \Delta |v|^2 dx \quad (2.26)$$

Now, we have the point-wise convergence

$$[Q(x) + \varepsilon]_-^\delta \rightarrow 1 \quad \forall x \in D_-^\varepsilon.$$

as  $\delta \downarrow 0$ . Hence, passing  $\delta \downarrow 0$  in (2.26), by the dominated convergence theorem, we obtain

$$\int_{D_-^\varepsilon} |\nabla v|^2 dx = \frac{1}{2} \int_{D_-^\varepsilon} \Delta |v|^2 dx, \quad (2.27)$$

which holds for all  $\varepsilon > 0$ . For a sequence  $\{\varepsilon_n\}$  with  $\varepsilon_n \downarrow 0$  as  $n \rightarrow \infty$ , we observe

$$D_-^{\varepsilon_n} \uparrow \cup_{n=1}^\infty D_-^{\varepsilon_n} = D_- := \{x \in \mathbb{R}^3 \mid Q(x) < 0\}.$$

Since

$$\begin{aligned} \left| \int_{\mathbb{R}^3} v \cdot \Delta v dx \right| &\leq \|v\|_{L^6} \|\Delta v\|_{L^{\frac{6}{5}}} \leq C \|\nabla v\|_{L^2} \|\Delta v\|_{L^{\frac{6}{5}}} \\ &\leq C \|\Delta v\|_{L^{\frac{6}{5}}}^2 < \infty, \end{aligned}$$

we have

$$\Delta |v|^2 = 2v \cdot \Delta v + 2|\nabla v|^2 \in L^1(\mathbb{R}^2). \quad (2.28)$$

Thus, we can apply the dominated convergence theorem in passing  $\varepsilon \downarrow 0$  in (2.27) to deduce

$$\int_{D_-} |\nabla v|^2 dx = \frac{1}{2} \int_{D_-} \Delta |v|^2 dx. \quad (2.29)$$

Now, thanks to (2.24) the set

$$S = \{x \in \mathbb{R}^3 \mid Q(x) = 0\}$$

consists of critical(maximum) points of  $Q$ , and hence  $\nabla Q(x) = 0$  for all  $x \in S$ , and the system (2.17) reduces to

$$-v \times \omega = \Delta v \quad \text{on} \quad S. \quad (2.30)$$

Multiplying (2.30) by  $v$ , we have that

$$0 = v \cdot \Delta v = \frac{1}{2} \Delta |v|^2 - |\nabla v|^2 \quad \text{on } S.$$

Therefore, one can extend the domain of integration in (2.29) from  $D_-$  to  $D_- \cup S = \mathbb{R}^3$ , and therefore

$$\int_{\mathbb{R}^3} |\nabla v|^2 dx = \frac{1}{2} \int_{\mathbb{R}^3} \Delta |v|^2 dx. \quad (2.31)$$

We now claim the right hand side of (2.31) vanishes. Since  $\Delta |v|^2 \in L^1(\mathbb{R}^3)$  from (2.28), applying the dominated convergence theorem, we have

$$\begin{aligned} \left| \int_{\mathbb{R}^3} \Delta |v|^2 dx \right| &= \lim_{R \rightarrow \infty} \left| \int_{\mathbb{R}^3} \Delta |v|^2 \sigma_R dx \right| = \lim_{R \rightarrow \infty} \left| \int_{\mathbb{R}^3} |v|^2 \Delta \sigma_R dx \right| \\ &\leq \lim_{R \rightarrow \infty} \int_{\mathbb{R}^3} |v|^2 |\Delta \sigma_R| dx \\ &\leq \lim_{R \rightarrow \infty} \frac{\|D^2 \sigma\|_{L^\infty}}{R^2} \|v\|_{L^6(R \leq |x| \leq 2R)}^2 \left( \int_{\{R \leq |x| \leq 2R\}} dx \right)^{\frac{2}{3}} \\ &\leq C \|D^2 \sigma\|_{L^\infty} \lim_{R \rightarrow \infty} \|v\|_{L^6(R \leq |x| \leq 2R)}^2 = 0 \end{aligned}$$

as claimed. Thus (2.31) implies that

$$\nabla v = 0 \quad \text{on } \mathbb{R}^3,$$

and  $v = \text{constant}$ . By (1.10) we have  $v = 0$ .  $\square$

*Remark after the proof of Theorem 1.3:* The first part of the above proof, showing  $[Q]_+ = 0$  can be also done by applying the maximum principle, which is from the following identity for  $Q$ ,

$$-\Delta Q + v \cdot \nabla Q = -|\omega|^2 \leq 0$$

I do not think, however, the maximum principle can also be applied to the proof of the second part, showing  $[Q]_- = 0$ , which is more subtle than the first part. The above proof overall shows that the argument of the proof I used for this second part can also be adapted for the first part without using the maximum principle, which exhibits consistency.

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